On the Summation of Progressions by means of infinite Series *

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§140 The general expression, we found in the preceding chapter for the summatory term of a series, whose general term or the term corresponding to the index x is $= z$, i.e.

$$Sz = \int zdx + \frac{1}{2}z + \frac{\mathfrak{A}dz}{1 \cdot 2dx} - \frac{\mathfrak{B}d^3z}{1 \cdot 2 \cdot 3 \cdot 4dx^3} + \frac{\mathfrak{C}d^5z}{1 \cdot 2 \cdots 6dx^5} - \text{etc.},$$

especially yields the summation of series whose general terms are polynomial functions of the index x, since in these cases one eventually gets to vanishing differentials. But if z was not a function of x of such a kind, then its differentials proceed to infinity and so an infinite series expressing the sum of the propounded series up to the given term, whose index is $= x$, results. Therefore, the sum of the propounded series continued to infinity will result for $x = \infty$; and this way another infinite series equal to the first arises.

§141 But if one puts $x = 0$, then the expression exhibiting the sum has to vanish, as we already mentioned; if this does not happen, a constant quantity has either to be added or subtracted that this condition is satisfied. If, having done this, one puts $x = 1$, the found sum will yield the first term of the series; but if one puts $x = 2$, the aggregate of the first and the second, if $x = 3$,

*Original title: "De Summatione Progressionum per Series infinitas", first published as part of the book *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, 1755, reprinted in in *Opera Omnia*: Series 1, Volume 10, pp. 337 - 367 ", Eneström-Number E212, translated by: Alexander Aycock for the „Euler-Kreis Mainz" the aggregate of the three initial terms of the series will result, and so forth. Therefore, in these cases, since the sum of one or two or three etc. terms is known, the value of the infinite series expressing this sum will become known, and from this source one will be able to sum innumerable series.

§142 Since, if a constant of such a kind was added to the sum that it vanishes for $x = 0$, the sum is correct in the remaining cases, i.e. for all x, it is obvious, if a constant quantity of such a kind is added to the found sum that in one special case the true sum results, that then the true sum has to result also in all remaining cases. Hence, if having put put $x = 0$ it is not clear, a value of which kind the expression of the sum will have, and hence the constant to be added cannot be found, then any other number can be assumed for x and by adding a constant the correct sum can be forced to result; how this has to happen, will become more perspicuous in the following.

§142[a] At first, let us consider this harmonic progression

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x} = s;$$

because its general term is $= \frac{1}{x}$, it will be $z = \frac{1}{x}$ and the summatory term s will be found this way. First, it will be $\int z dx = \int \frac{dx}{x} = \log x$; further, the differentials will be as follows

$$\frac{dz}{dx} = -\frac{1}{x^2}, \quad \frac{ddz}{2dx^2} = \frac{1}{x^3}, \quad \frac{d^3z}{6dx^3} = -\frac{1}{x^4}, \quad \frac{d^4z}{24dx^4} = \frac{1}{x^5}, \quad \frac{d^5z}{120dx^5} = -\frac{1}{x^6} \quad \text{etc.}$$

Therefore, it will be

$$s = \log x + \frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \text{etc.} + \text{Constant.}$$

Therefore, the constant to be added here cannot be defined from the case $x = 0$. So put $x = 1$, since then $s = 1$; it will be

$$1 = \frac{1}{2} - \frac{\mathfrak{A}}{2} + \frac{\mathfrak{B}}{4} - \frac{\mathfrak{C}}{6} + \frac{\mathfrak{D}}{8} - \text{etc.} + \text{Const.},$$

whence this constant becomes

$$= \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \text{etc.},$$

and hence the summatory term in question will be

$$s = \log x + \frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \text{etc.}$$
$$+ \frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{C}}{8} + \text{etc.}$$

§143 Since the Bernoulli numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. constitute a divergent series, this value of the constant cannot be found. But if a larger number is substituted for x and the sum of that many terms is actually found, the value of the constant can be investigated conveniently. For this purpose, set $x = 10$ and by adding the first ten terms, one will find their sum to be

$$= 2.9289682539682539\,68,$$

to which the expression for the sum has to be equal, if one puts $x = 10$; that expression is

$$\log 10 + \frac{1}{20} - \frac{\mathfrak{A}}{200} + \frac{\mathfrak{B}}{40000} - \frac{\mathfrak{C}}{6000000} + \frac{\mathfrak{D}}{800000000} - \text{etc.} + C.$$

Therefore, having taken the hyperbolic logarithm of ten for $\log 10$ and having substituted the values found above [§ 122] for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc., one will find that constant to be

$$C = 0.5772156649015325,$$

which number therefore is equal to the sum of the series

$$\frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \frac{\mathfrak{E}}{10} - \text{etc.}$$

§144 If sufficiently small numbers are substituted for x, since the sum of the series is actually easily found, one will obtain the sum of this series

$$\frac{1}{2x} - \frac{\mathfrak{A}}{2x^2} + \frac{\mathfrak{B}}{4x^4} - \frac{\mathfrak{C}}{6x^6} + \frac{\mathfrak{D}}{8x^8} - \text{etc.} = s - \log x - C.$$

But if x denotes a very large number, since then the value of this infinite expression is easily assigned in decimal numbers, vice versa the sum of the series will be defined. And first it is certainly clear, if the series is continued to infinity, that its sum will be infinitely large; for, having put $x = \infty$, also $\log x$ becomes infinite, even though $\log x$ has an infinitely small ratio to x. But in order to assign the sum of an arbitrary number of terms of the series in a convenient manner, let us express the values of the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. in decimal fractions.

$$\begin{aligned}
\mathfrak{A} &= 0.1666666666666 \\
\mathfrak{B} &= 0.0333333333333 \\
\mathfrak{C} &= 0.0238095238095 \\
\mathfrak{D} &= 0.0333333333333 \\
\mathfrak{E} &= 0.0757575757575 \\
\mathfrak{F} &= 0.2531135531135 \\
\mathfrak{G} &= 1.1666666666666 \\
\mathfrak{H} &= 7.0921568627451 \quad \text{etc.}
\end{aligned}$$

whence

$$\begin{aligned}
\frac{\mathfrak{A}}{2} &= 0.0833333333333 \\
\frac{\mathfrak{B}}{4} &= 0.0083333333333 \\
\frac{\mathfrak{C}}{6} &= 0.0039682539682 \\
\frac{\mathfrak{D}}{8} &= 0.0041666666666 \\
\frac{\mathfrak{E}}{10} &= 0.0075757575757 \\
\frac{\mathfrak{F}}{12} &= 0.0210927960928 \\
\frac{\mathfrak{G}}{14} &= 0.0833333333333 \\
\frac{\mathfrak{H}}{16} &= 0.4432598039316
\end{aligned}$$

EXAMPLE 1

To find the sum of thousand terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} +$ *etc.*

Therefore, put $x = 1000$, and since

$$\log 10 \quad = 2.302585092994045684 0,$$

it will be

$$\begin{aligned}
\log x \quad &= 6.9077553789821 \\
\text{Const.} \quad &= 0.5772156649015 \\
\frac{1}{2x} \quad &= 0.0005000000000
\end{aligned}$$

and in total

$$\begin{aligned}
&= 7.4854709438836 \\
\text{subtr. } \frac{\mathfrak{A}}{2xx} &= 0.0000000833333
\end{aligned}$$

which yields

$$\begin{aligned}
&= 7.4854708605503 \\
\text{add } \frac{\mathfrak{B}}{4x^4} &= 0.0000000000000
\end{aligned}$$

Therefore

$$= 7.4854708605503$$

is the sum of thousand terms in question, which is still smaller than seven and a half units.

EXAMPLE 2

To find the sum of a million terms of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} +$ *etc.*

Since $x = 1000000$, it will be $\log x = 6 \cdot \log 10$, therefore

$$\begin{aligned}
\log x \quad &= 12.8155105579642 \\
\text{Const.} \quad &= \quad 0.05772156649015 \\
\frac{1}{2x} \quad &= \quad 0.0000005000000
\end{aligned}$$

in total

$$= \quad 14.3927267228657 = \text{the sum in question}$$

§145 Therefore, if one substitutes a very large number for x, the sum is found to a high degree of accuracy using only the first term $\log x$ increased by the constant C; therefore, extraordinary corollaries can be deduced from this. So, if x was a very large number and one puts

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{x}=s$$

and

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{x}+\cdots+\frac{1}{x+y}=t,$$

since approximately $s=\log x+C$ and $t=\log(x+y)+C$, it will be

$$t-s=\log(x+y)-\log x=\frac{x+y}{x}$$

and hence this logarithm is approximately expressed by means of a harmonic series consisting of a finite numbers of terms as follows

$$\log\frac{x+y}{y}=\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\cdots+\frac{1}{x+y}.$$

But this logarithm is exhibited more accurately, if more terms of the above sums s and t are taken . So, since

$$s=\log x+C+\frac{1}{2x}-\frac{1}{12xx} \quad \text{and} \quad t=\log(x+y)+C+\frac{1}{2(x+y)}-\frac{1}{12(x+y)^2}$$

it will be

$$t-s=\log\frac{x+y}{x}-\frac{1}{2x}+\frac{1}{2(x+y)}+\frac{1}{12xx}-\frac{1}{12(x+y)^2},$$

and hence

$$\log\frac{x+y}{x}=\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\cdots+\frac{1}{x+y}+\frac{1}{2x}-\frac{1}{2(x+y)}-\frac{1}{12xx}+\frac{1}{12(x+y)^2}.$$

But if x is such a large number that the two last terms can be neglected, it will approximately be

$$\log\frac{x+y}{x}=\frac{1}{x+1}+\frac{1}{x+2}+\frac{1}{x+3}+\cdots+\frac{1}{x+y}+\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+y}\right).$$

§ 145[a] Using this harmonic series, we will also be able to define the sum of this series, in which only the odd numbers occur, i.e.

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots+\frac{1}{2x+1}.$$

For, since, taking all terms,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{2x}+\frac{1}{2x+1}$$

$$=\log(2x+1)+C+\frac{1}{2(2x+1)}-\frac{\mathfrak{A}}{2(2x+1)^2}+\frac{\mathfrak{B}}{4(2x+1)^4}-\frac{\mathfrak{C}}{6(2x+1)^6}+\text{etc.},$$

the sum of the even terms

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2x}$$

is the half of the above series, i.e.

$$\frac{1}{2}C+\frac{1}{2}\log x+\frac{1}{4x}-\frac{\mathfrak{A}}{4x^2}+\frac{\mathfrak{B}}{8x^4}-\frac{\mathfrak{C}}{12x^6}+\frac{\mathfrak{D}}{16x^8}-\text{etc.},$$

having subtracted this series from the latter,

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2x+1}$$

$$=\frac{1}{2}C+\log\frac{2x+1}{\sqrt{x}}+\frac{1}{2(2x+1)}-\frac{\mathfrak{A}}{2(2x+1)^2}+\frac{\mathfrak{B}}{4(2x+1)^4}-\text{etc.}$$

$$-\frac{1}{4x}+\frac{\mathfrak{A}}{4x^2}-\frac{\mathfrak{B}}{8x^4}+\text{etc.}$$

§146 One can indeed even find the sum of any harmonic series by means of the same general expression; for, let

$$\frac{1}{m+n}+\frac{1}{2m+n}+\frac{1}{3m+n}+\frac{1}{4m+n}+\cdots+\frac{1}{mx+n}=s;$$

since the general term is $z=\frac{1}{mx+n}$, it will be

$$\int zdx=\frac{1}{m}\log(mx+n),\quad \frac{dz}{dx}=-\frac{m}{(mx+n)^2},\quad \frac{ddz}{2dx^2}=\frac{mm}{(mx+n)^3},$$

$$\frac{d^3z}{6dx^3}=-\frac{m^3}{(mx+n)^4},\quad \frac{d^4z}{24dx^4}=\frac{m^4}{(mx+n)^5},\quad \frac{d^5z}{120dx^5}=-\frac{m^5}{(mx+n)^6}\quad \text{etc.}$$

From these one hence finds

$$s=D+\frac{1}{m}\log(mx+n)+\frac{1}{2(mx+n)}-\frac{\mathfrak{A}m}{2(mx+n)^2}+\frac{\mathfrak{B}m^3}{4(mx+n)^4}$$

$$-\frac{\mathfrak{C}m^5}{6(mx+n)^6}+\frac{\mathfrak{D}m^7}{8(mx+n)^8}-\text{etc.}$$

Therefore, having put $x=0$, the constant to be added will be

$$D=-\frac{1}{m}\log n-\frac{1}{2n}+\frac{\mathfrak{A}m}{2n^2}-\frac{\mathfrak{B}m^3}{4n^4}+\frac{\mathfrak{C}m^5}{6n^6}-\text{etc.}$$

§147 But if $n=0$, since the sum of the series

$$\frac{1}{m}+\frac{1}{2m}+\frac{1}{3m}+\frac{1}{4m}+\cdots+\frac{1}{mx}$$

is

$$=\frac{1}{m}C+\frac{1}{m}\log x+\frac{1}{2mx}-\frac{\mathfrak{A}}{2mx^2}+\frac{\mathfrak{B}}{4mx^4}-\text{etc.},$$

but the sum of this series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{mx}$$

is

$$= C + \log mx + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \text{etc.},$$

if from this series the other series is subtracted m times that this series results

$$\begin{array}{l} 1 + \frac{1}{2} + \cdots + \frac{1}{m} + \cdots + + \frac{1}{2m} + \cdots + \frac{1}{3m} + \cdots + \frac{1}{mx} \\ \qquad\qquad - \frac{m}{m} \qquad\qquad - \frac{m}{2m} \qquad - \frac{m}{3m} \qquad\quad - \frac{m}{mx}, \end{array}$$

its sum will be

$$\begin{aligned} &= \log m + \frac{1}{2mx} - \frac{\mathfrak{A}}{2m^2x^2} + \frac{\mathfrak{B}}{4m^4x^4} - \text{etc.} \\ &\qquad\quad - \frac{1}{2x} + \frac{\mathfrak{A}}{2xx} - \frac{\mathfrak{B}}{4x^4} + \text{etc.}, \end{aligned}$$

and if one sets $x = \infty$, the sum will be $= \log m$. Hence, by taking the numbers 2, 3, 4 etc. for m it will be

$$\begin{aligned} \log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \text{etc.} \\ \log 3 &= 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \text{etc.} \\ \log 4 &= 1 + \frac{1}{2} + \frac{1}{3} - \frac{3}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} - \frac{3}{8} + \text{etc.} \\ \log 5 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{4}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{4}{10} + \text{etc.} \\ &\text{etc.} \end{aligned}$$

§148 Having discussed the harmonic series now, let us proceed to the reciprocal series of the squares and let

$$s = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{xx};$$

since its general term is $z = \frac{1}{xx}$, it will be $\int z dx = -\frac{1}{x}$ and the differentials of z will be

$$\frac{dz}{2dx} = -\frac{1}{x^3}, \quad \frac{ddz}{2\cdot 3dx^2} = \frac{1}{x^4}, \quad \frac{d^3z}{2\cdot 3\cdot 4dx^3} = -\frac{1}{x^5} \quad \text{etc.},$$

whence the sum will be

$$s = C - \frac{1}{x} + \frac{1}{2xx} - \frac{\mathfrak{A}}{x^3} + \frac{\mathfrak{B}}{x^5} - \frac{\mathfrak{C}}{x^7} + \frac{\mathfrak{D}}{x^9} - \frac{\mathfrak{E}}{x^{11}} + \text{etc.},$$

in which the constant C to be added is to be defined from a special case, in which the sum is known. Therefore, let us put $x = 1$; since $x = 1$, it has to be

$$C = 1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \text{etc.},$$

which series, since highly divergent, does not show the value of the constant C. But since we demonstrated above [§ 125] that the sum of this series continued to infinity is $= \frac{\pi\pi}{6}$, for $x = \infty$, if one puts $s = \frac{\pi\pi}{6}$, it will be $C = \frac{\pi\pi}{6}$ because all remaining terms vanish. Therefore, it will be

$$1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \mathfrak{E} - \text{etc.} = \frac{\pi\pi}{6}.$$

§149 But if the sum of this series would have not been known, the value of that constant C would have to be determined from another case, in which the sum was actually found. For this aim, let us put $x = 10$ and actually adding ten terms one will find

$$s = 1.549767731166540690$$

then

$$\begin{array}{lll}
\text{add} & \dfrac{1}{x} & = 0.1 \\
\text{subtr.} & \dfrac{1}{2xx} & = 0.0005 \\
\hline
& & 1.644767731166540690 \\
\text{add} & \dfrac{\mathfrak{A}}{x^3} & = 0.000166666666666666 \\
& & 1.644934397833207356 \\
\hline
\text{subtr.} & \dfrac{\mathfrak{B}}{x^5} & = 0.000000333333333333 \\
& & 1.644934064499874023 \\
\hline
\text{add} & \dfrac{\mathfrak{C}}{x^7} & = 0.000000002380952381 \\
& & 1.644934066880826404 \\
\hline
\text{subtr.} & \dfrac{\mathfrak{D}}{x^9} & = 0.000000000033333333 \\
& & 1.6444066847493071 \\
\hline
\end{array}$$

$$\text{add} \quad \frac{\mathfrak{E}}{x^{11}} = 0.000000000000757575$$

$$1.644934066848250646$$

$$\text{subtr.} \quad \frac{\mathfrak{F}}{x^{13}} = 0.000000000000025311$$

$$1.644934066848225335$$

$$\text{add} \quad \frac{\mathfrak{G}}{x^{15}} = 0.000000000000001166$$

$$\text{subtr.} \quad \frac{\mathfrak{H}}{x^{17}} = 0.000000000000000071$$

$$1.644934066848226430 = C.$$

And this value at the same time is the value of the expression $\frac{\pi\pi}{6}$, as anyone carrying out the calculation using the known value of π will discover. Therefore, it is understood at the same time, even though the series $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. diverges, that nevertheless the true sum results this way.

§150 Now let $z = \frac{1}{x^3}$ and

$$s = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{x^3};$$

since

$$\int z dx = -\frac{1}{2xx}, \quad \frac{dz}{1 \cdot 2 \cdot 3 dx} = -\frac{1}{2x^4}, \quad \frac{ddz}{1 \cdot 2 \cdot 3 \cdot 4 dx^2} = \frac{1}{2x^5},$$
$$\frac{d^3 z}{1 \cdot 2 \cdots 5 dx^3} = -\frac{1}{2x^6}, \quad \frac{d^4 z}{1 \cdot 2 \cdots 6 dx^4} = \frac{1}{2x^7}, \quad \frac{d^5 z}{1 \cdot 2 \cdots 7 dx^5} = -\frac{1}{2x^6} \quad \text{etc.},$$

it will be

$$s = C - \frac{1}{2xx} + \frac{1}{2x^3} - \frac{3\mathfrak{A}}{2x^4} + \frac{5\mathfrak{B}}{2x^6} - \frac{7\mathfrak{C}}{2x^6} + \text{etc.}$$

and hence, having put $x = 1$, because of $s = 1$, it will be

$$C = 1 + \frac{1}{2} - \frac{1}{2} + \frac{3}{2}\mathfrak{A} - \frac{5}{2}\mathfrak{B} + \frac{7}{2}\mathfrak{C} - \frac{9}{2}\mathfrak{D} + \text{etc.}.$$

and this value of C at the same time will show the sum of the propounded series, if it is continued to infinity. Since the sums of the odd powers are not known as the sum of the even powers, this value of C has to be defined from the known sum of some terms. Therefore, let $x = 10$; it will be

$$C = s + \frac{1}{2xx} - \frac{1}{2x^3} + \frac{2\mathfrak{A}}{2x^4} - \frac{5\mathfrak{B}}{2x^6} + \frac{7\mathfrak{C}}{2x^8} - \text{etc.}$$

But, in order to perform the calculation more easily, note that

$$\begin{aligned}
\frac{3\mathfrak{A}}{2} &= 0.250000000000 \\
\frac{5\mathfrak{B}}{2} &= 0.833333333333 \\
\frac{7\mathfrak{C}}{2} &= 0.833333333333 \\
\frac{9\mathfrak{D}}{2} &= 0.150000000000 \\
\frac{11\mathfrak{E}}{2} &= 0.416666666666 \\
\frac{13\mathfrak{F}}{2} &= 1.645280952380 \\
\frac{15\mathfrak{G}}{2} &= 8.750000000000 \\
\frac{17\mathfrak{H}}{2} &= 60.283333333333 \\
&\text{etc.}
\end{aligned}$$

Therefore, the terms to be added to s will become

$$
\begin{aligned}
\frac{1}{2xx} &= 0.005000000000000000 \\
\frac{3\mathfrak{A}}{2x^4} &= 0.000025000000000000 \\
\frac{7\mathfrak{C}}{2x^8} &= 0.000000000833333333 \\
\frac{11\mathfrak{E}}{2x^{12}} &= 0.000000000000416666 \\
\frac{13\mathfrak{F}}{2x^{16}} &= 0.000000000000000875 \\
\hline
& 0.005025000833750875
\end{aligned}
$$

but the terms to be subtracted are

$$\frac{1}{2x^3} = 0.005000000000000000$$

$$\frac{5\mathfrak{B}}{2x^6} = 0.000000000833333333$$

$$\frac{9\mathfrak{D}}{2x^{10}} = 0.000000000001500000$$

$$\frac{13\mathfrak{F}}{2x^{14}} = 0.000000000000016452$$

$$\frac{17\mathfrak{F}}{2x^{18}} = 0.000000000000000060$$

$$0.000500083348349845$$

$$\text{from} \quad 0.005025000833750875$$

$$0.004524917485401030$$

$$s = 1.197531985674193251$$

$$C = 1.202056903159594281$$

§151 If we continue this way, we will find the sum of all series of reciprocal powers expressed in decimal fractions.

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc.}=1.6449340668482264=\frac{2\mathfrak{A}}{1\cdot 2}\pi^2$$

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc.}=1.2020569031595942$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}=1.0823232337111381=\frac{2^3\mathfrak{B}}{1\cdot 2\cdot 3\cdot 4}\pi^4$$

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc.}=1.0369277551433699$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=1.0173430619844491=\frac{2^5\mathfrak{C}}{1\cdot 2\cdots 6}\pi^6$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\text{etc.}=1.0083492773819288$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc.}=1.0040773561979443=\frac{2^7\mathfrak{D}}{1\cdot 2\cdots 8}\pi^8$$

$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\text{etc.}=1.0020083928260822$$

$$1+\frac{1}{2^{10}}+\frac{1}{3^{10}}+\frac{1}{4^{10}}+\text{etc.}=1.0009945751278180=\frac{2^9\mathfrak{E}}{1\cdot 2\cdots 10}\pi^{10}$$

$$1+\frac{1}{2^{11}}+\frac{1}{3^{11}}+\frac{1}{4^{11}}+\text{etc.}=1.0004941886041194$$

$$1+\frac{1}{2^{12}}+\frac{1}{3^{12}}+\frac{1}{4^{12}}+\text{etc.}=1.0002460865533080=\frac{2^{11}\mathfrak{F}}{1\cdot 2\cdots 12}\pi^{12}$$

$$1+\frac{1}{2^{13}}+\frac{1}{3^{13}}+\frac{1}{4^{13}}+\text{etc.}=1.0001227133475784$$

$$1+\frac{1}{2^{14}}+\frac{1}{3^{14}}+\frac{1}{4^{14}}+\text{etc.}=1.0000612481350587=\frac{2^{13}\mathfrak{G}}{1\cdot 2\cdots 14}\pi^{14}$$

$$1+\frac{1}{2^{15}}+\frac{1}{3^{15}}+\frac{1}{4^{15}}+\text{etc.}=1.0000305882363070$$

$$1+\frac{1}{2^{16}}+\frac{1}{3^{16}}+\frac{1}{4^{16}}+\text{etc.}=1.0000152822594086=\frac{2^{15}\mathfrak{H}}{1\cdot 2\cdots 16}\pi^{16}$$

etc.

§152 From these vice versa the sums of those series consisting of the Bernoulli numbers can be exhibited. For, it will be

$$1 + -\frac{1}{2} + \frac{\mathfrak{A}}{2} - \frac{\mathfrak{B}}{4} + \frac{\mathfrak{C}}{6} - \frac{\mathfrak{D}}{8} + \text{etc.} = 0.57721 \text{ etc.}$$

$$1 + 1 - \frac{1}{2} + \mathfrak{A} - \mathfrak{B} + \mathfrak{C} - \mathfrak{D} + \text{etc.} = \frac{2\mathfrak{A}}{1 \cdot 2}\pi^2$$

$$1 + \frac{1}{2} - \frac{1}{2} + \frac{2\mathfrak{A}}{2} - \frac{5\mathfrak{B}}{2} + \frac{7\mathfrak{C}}{2} - \frac{9\mathfrak{D}}{2} + \text{etc.} = 1.2020 \text{ etc.}$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{3 \cdot 4\mathfrak{A}}{2 \cdot 3} - \frac{5 \cdot 5\mathfrak{B}}{2 \cdot 3} + \frac{7 \cdot 8\mathfrak{C}}{2 \cdot 3} - \frac{9 \cdot 10\mathfrak{D}}{2 \cdot 3} + \text{etc.} = \frac{2^3\mathfrak{B}}{1 \cdot 2 \cdot 3 \cdot 4}\pi^4$$

$$1 + \frac{1}{4} - \frac{1}{2} + \frac{3 \cdot 4 \cdot 5\mathfrak{A}}{2 \cdot 3 \cdot 4} - \frac{5 \cdot 6 \cdot 7\mathfrak{B}}{2 \cdot 3 \cdot 4} + \frac{7 \cdot 8 \cdot 9\mathfrak{C}}{2 \cdot 3 \cdot 4} - \frac{9 \cdot 10 \cdot 11\mathfrak{D}}{2 \cdot 3 \cdot 4} + \text{etc.} = 1.0369 \text{ etc.}$$

$$1 + \frac{1}{5} - \frac{1}{2} + \frac{3 \cdot 4 \cdot 5 \cdot 6\mathfrak{A}}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{4 \cdot 5 \cdot 6 \cdot 7\mathfrak{B}}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{5 \cdot 6 \cdot 7 \cdot 8\mathfrak{C}}{2 \cdot 3 \cdot 4 \cdot 5} - \text{etc.} = \frac{2^5\mathfrak{C}}{1 \cdot 2 \cdots 6}\pi^6$$

etc.

Therefore, each second of these series can be summed by means of the quadrature of the circle; it is not known to this day, on which transcendental quantity the remaining series depend; for, they cannot be reduced to powers of π with odd exponents, such that the coefficients would be rational numbers. But that this at least becomes approximately clear, of what nature the coefficients of powers of π will be for odd exponents, we added the following table.

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\text{etc. to infinity}=\frac{\pi}{0.0000}\quad=\infty$$

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\text{etc. to infinity}=\frac{\pi^2}{6.0000}\quad\text{exactly}$$

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\text{etc. to infinity}=\frac{\pi^3}{25.79436}\quad\text{approximately}$$

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc. to infinity}=\frac{\pi^2}{90.00000}\quad\text{exactly}$$

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\text{etc. to infinity}=\frac{\pi^5}{295.1215}\quad\text{approximately}$$

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc. to infinity}=\frac{\pi^6}{945.000}\quad\text{exactly}$$

$$1+\frac{1}{2^7}+\frac{1}{3^7}+\frac{1}{4^7}+\text{etc. to infinity}=\frac{\pi^2}{2995.284}\quad\text{approximately}$$

$$1+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\text{etc. to infinity}=\frac{\pi^2}{9450.0000}\quad\text{exactly}$$

$$1+\frac{1}{2^9}+\frac{1}{3^9}+\frac{1}{4^9}+\text{etc. to infinity}=\frac{\pi^9}{29749.35}\quad\text{approximately}$$

etc.

§153 From this source the series of Bernoulli numbers

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ \mathfrak{A}, & \mathfrak{B}, & \mathfrak{C}, & \mathfrak{D}, & \mathfrak{E}, & \mathfrak{F}, & \mathfrak{G}, & \mathfrak{H}, & \mathfrak{I} & \text{etc.,} \end{array}$$

even though it seems to be rather irregular, can be interpolated, i.e. the terms lying in the middle between any two consecutive ones can be assigned; for, if the term falling in the middle between the first $\mathfrak{A}$ and the second $\mathfrak{B}$ or the one corresponding to the index $1\frac{1}{2}$ was $=p$, it will be

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\text{etc.}=\frac{2^2p}{1\cdot 2\cdot 3}\pi^3$$

and hence

$$p=\frac{3}{2\pi^3}\left(1+\frac{1}{2^3}+\frac{1}{3^3}+\text{etc.}\right)=0.05815227.$$

If in like manner the term falling in the middle between $\mathfrak{B}$ and $\mathfrak{C}$ or the one corresponding to the index $2\frac{1}{2}$ is put $=q$, since it will be

$$1+\frac{1}{2^5}+\frac{1}{3^5}+\text{etc.}=\frac{2^4q}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5,$$

it will also be

$$q=\frac{15}{2\pi^5}\left(1+\frac{1}{2^5}+\frac{1}{3^5}+\text{etc.}\right)=0.02541327.$$

Therefore, if the sums of these series, in which the exponents of the powers are odd numbers, could be exhibited, then also the series of the Bernoulli numbers could be interpolated.

§154 Now, let us put $z=\frac{1}{nn+xx}$ and find the sum of this series

$$s=\frac{1}{nn+1}+\frac{1}{nn+4}+\frac{1}{nn+9}+\cdots+\frac{1}{nn+xx}.$$

Since $\int zdx=\int\frac{dx}{nn+xx}$, it will be

$$\int zdx=\frac{1}{n}\arctan\frac{x}{n}.$$

Put $\operatorname{arccot}\frac{x}{n}=u$; it will be

$$\int zdx=\frac{1}{n}\left(\frac{\pi}{2}-u\right)$$

and

$$\frac{x}{n}=\cot u=\frac{\cos u}{\sin u}\quad\text{and}\quad\frac{nn+xx}{nn}=\frac{1}{\sin^2 u}\quad\text{and}\quad z=\frac{\sin^2 u}{nn}\quad\text{and}\quad\frac{dx}{n}=-\frac{du}{\sin^2 u},$$

whence

$$du = -\frac{dx\sin^2 u}{n}.$$

Therefore, the differentials of z will be found this way

$$dz = \frac{2du\sin u\cdot\cos u}{nn} = -\frac{dx\sin^2 u\sin 2u}{n^3} \quad \text{and} \quad \frac{dz}{dx} = -\frac{\sin^2 u\cdot\sin 2u}{n^3},$$

$$\frac{ddz}{dx^2} = -\frac{du(\sin u\cdot\cos u\cdot\sin 2u + \sin^2 u\cdot\cos 2u)}{n^3} = \frac{dx\sin^3\cdot 3u}{n^4}$$

and

$$\frac{ddz}{2dx^2} = \frac{\sin^3 u\cdot\sin 3u}{n^4}.$$

In the same way, as we already found above [§ 87] for the same case, it will be

$$\frac{d^3z}{2\cdot 3dx^3} = -\frac{\sin^4\cdot\sin 4u}{n^5}, \quad \frac{d^4z}{2\cdot 3\cdot 4dx^4} = \frac{\sin^5\cdot\sin 5u}{n^6} \quad \text{etc.},$$

from which the sum in question will be formed

$$s = \frac{\pi}{2n} - \frac{u}{n} + \frac{\sin u\cdot\sin u}{2nn} - \frac{\mathfrak{A}}{2}\cdot\frac{\sin^2 u\cdot\sin 2u}{n^3} + \frac{\mathfrak{B}}{4}\cdot\frac{\sin^4 u\cdot\sin 4u}{n^5}$$
$$-\frac{\mathfrak{C}}{6}\cdot\frac{\sin^6 u\cdot\sin 6u}{n^7} + \frac{\mathfrak{D}}{8}\cdot\frac{\sin^8 u\cdot\sin 8u}{n^9} - \text{etc.} + \text{Const.}$$

If here in order to determine the constant one sets $x = 0$, in which case $s = 0$, it will be $\cot u = 0$ and hence u the angle of 90° and therefore $\sin u = 1$, $\sin 2u = 0$, $\sin 4u = 0$, $\sin 6u = 0$ etc.; therefore, it seems that

$$0 = \frac{\pi}{2n} - \frac{\pi}{2n} + \frac{1}{2nn} + C \quad \text{and hence} \quad C = -\frac{1}{2nn};$$

but on the other hand it is to be noted, even though the remaining terms vanish, that nevertheless, since the coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. eventually grow to infinity, their sum can be finite.

§155 To determine this constant correctly, let us put that $x = \infty$; for, we defined the sum of this series running to infinity already in the *Introdctio* and showed it to be

$$= -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2\pi b} - 1)}.$$

But having put $x = \infty$, it will be $u = 0$ and hence $\sin u = 0$ and at the same time the sines of all multiple arcs will vanish. But since in this series the powers of $\sin u$ grow, the divergence of the series is no obstruction for the vanishing of the value in this case. Therefore, it will become $s = \frac{\pi}{2n} + C$; hence, it will be

$$\frac{\pi}{2n} + C = -\frac{1}{2nn} + \frac{\pi}{2n} + \frac{\pi}{n(e^{2n\pi} - 1)} \quad \text{and} \quad C = -\frac{1}{2nn} + \frac{\pi}{n(e^{2n\pi} - 1)}.$$

Therefore, the sum of the series in question will be

$$\begin{aligned} s = \frac{\pi}{2n} - \frac{u}{n} - \frac{1}{2nn} + \frac{\sin^2 u}{2nn} - \frac{\mathfrak{A}}{2} \cdot \frac{\sin^2 u \cdot \sin 2u}{n^3} \\ + \frac{\mathfrak{B}}{4} \cdot \frac{\sin^4 u \cdot \sin 4u}{n^5} - \frac{\mathfrak{C}}{6} \cdot \frac{\sin^6 u \cdot \sin 6u}{n^7} + \text{etc.} + \frac{\pi}{n(e^{2n\pi} - 1)}. \end{aligned}$$

Where it is to be noted, if n was a modestly large number, that the last term $\frac{\pi}{n(e^{2\pi n} - 1)}$ will become so small that it can be neglected.

§156 Let us put $x = n$ so that

$$s = \frac{1}{nn+1} + \frac{1}{nn+4} + \frac{1}{nn+9} + \cdots + \frac{1}{nn+nn}.$$

Then it will be $\cot u = 1$ and $u = 45^\circ = \frac{\pi}{4}$. Therefore, one will have $\sin u = \frac{1}{\sqrt{2}}$, $\sin 2u = 1$, $\sin 4u = 0$, $\sin 6u = -1$, $\sin 8u = 0$, $\sin 10u = 1$ etc. Therefore, it will be

$$s = \frac{\pi}{4n} - \frac{1}{2nn} + \frac{1}{4nn} - \frac{\mathfrak{A}}{2 \cdot 2n^3} + \frac{\mathfrak{C}}{6 \cdot 8n^7} - \frac{\mathfrak{E}}{10 \cdot 2^5 n^{11}} + \frac{\mathfrak{G}}{14 \cdot 2^7 n^{15}} - \text{etc.} + \frac{\pi}{n(e^{2n\pi} - 1)},$$

in which expression only each second Bernoulli number appears. Therefore, if the value of s was already found by actual calculation, hence the quantity π can be defined; for, it will be

$$\pi = 4ns + \frac{1}{n} + \frac{\mathfrak{A}}{1 \cdot n^2} - \frac{\mathfrak{C}}{3 \cdot 2^2 n^6} + \frac{\mathfrak{E}}{5 \cdot 2^4 n^{10}} - \frac{\mathfrak{G}}{7 \cdot 2^6 n^{14}} + \text{etc.} - \frac{4\pi}{e^{2n\pi} - 1}.$$

For, even though the last term contains π, nevertheless, since it is so small, it suffices to determine the value of π approximately.

EXAMPLE

Let $n = 5$; it will be

$$s = \frac{1}{26} + \frac{1}{29} + \frac{1}{34} + \frac{1}{41} + \frac{1}{50};$$

these terms, if they are actually added, will give

$$s = 0.146746306590549494;$$

Hence the terms will be

$$
\begin{aligned}
4ns &= 2.93492611381098988 \\
\frac{1}{n} &= 0.20000000000000000 \\
\frac{\mathfrak{A}}{nn} &= 0.00666666666666666 \\
&\overline{} \\
&\ 3.14159278047765654 \\
\frac{\mathfrak{C}}{3\cdot 2^2\cdot n^6} &= 0.00000012698412698 \\
&\overline{} \\
&\ 3.14159265349352956 \\
\frac{\mathfrak{D}}{5\cdot 2^4\cdot n^{10}} &= 0.00000000009696969 \\
&\overline{} \\
&\ 3.14159265359049925 \\
\frac{\mathfrak{E}}{7\cdot 2^6\cdot n^{14}} &= 0.00000000000042666 \\
&\overline{} \\
&\ 3.14159265359007259 \\
\frac{\mathfrak{F}}{9\cdot 2^8\cdot n^{18}} &= 0.00000000000000625 \\
&\overline{} \\
&\ 3.14159265359007884.
\end{aligned}
$$

This value already comes so close to the true value that one has to wonder why by such a simple calculation one can get this far. This expression is indeed a little bit larger than the correct value; for, one has still to subtract $\frac{4\pi}{e^{2n\pi}-1}$, whose value, as long as π is sufficiently accurately known, can be exhibited; this will be achieved by means of logarithms.

Since $\pi \log e = 1.3643763538$, it will be

$$\log e^{2n\pi} = 10\pi \log e = 13.6437635.$$

Since

$$\frac{4\pi}{e^{2n\pi}-1} = \frac{4\pi}{e^{2n\pi}} + \frac{4\pi}{e^{4n\pi}} + \text{etc.},$$

it is easily understood that for our calculation it suffices to have taken the first term. Therefore, let us increase the characteristic by the number 17, since we have the same number of decimal places; it will be

$$\begin{array}{llcr}
 & \log \pi & = & 17.4971498 \\
 & \log 4 & = & \underline{0.6020600} \\
 & & & 18.0992098 \\
 & \text{subtr. } \log e^{2n\pi} & = & 13.6437635 \\
\text{Therefore} & & & \underline{4.4554463} \\
 & \dfrac{4\pi}{e^{2n\pi}} & = & 28539 \\
\text{subtract from} & & & \\
 & & & \underline{3.14159265369007884} \\
\text{it will be} & \pi & = & 3.14159265358979345
\end{array}$$

which expression deviates from the true value just at the penultimate digit; this is not surprising, since we would have to subtract the term $\frac{\mathfrak{L}}{11 \cdot 2^{10} \cdot n^{22}}$, which gives 22, and so not even the last figure would have been wrong. Moreover, it is understood, if we would have taken a larger number for n, e.g. 10, that the circumference π could have been found up to 25 and more digits very easily.

§157 Now let us also substitute transcendental functions of x for z and let be $z = \log x$ taking hyperbolic logarithms, since ordinary ones are easily reduced to them, and let

$$s = \log 1 + \log 2 + \log 3 + \log 4 + \cdots + \log x.$$

Therefore, since $z = \log x$, it will be

$$\int z dx = x \log x - x;$$

for, its differential gives $dx \log x$. Furthermore,

$$\frac{dz}{dx} = \frac{1}{x}, \quad \frac{ddz}{dx^2} = -\frac{1}{x^2}, \quad \frac{d^3z}{1\cdot 2dx^3} = \frac{1}{x^3}, \quad \frac{d^4z}{1\cdot 2\cdot 3dx^4} = -\frac{1}{x^4}, \quad \frac{d^5z}{1\cdot 2\cdot 3\cdot 4dx^5} = \frac{1}{x^5} \quad \text{etc.}$$

Therefore, one will conclude that

$$s = x\log x - x + \frac{1}{2}\log x + \frac{\mathfrak{A}}{1\cdot 2x} - \frac{\mathfrak{B}}{3\cdot 4x^3} + \frac{\mathfrak{C}}{5\cdot 6x^5} - \frac{\mathfrak{D}}{7\cdot 8x^7} + \text{etc.} + \text{Const.}$$

But this constant, putting $x = 1$, since $s = \log 1 = 0$, will be defined as follows

$$C = 1 - \frac{\mathfrak{A}}{1\cdot 2} + \frac{\mathfrak{B}}{3\cdot 4} - \frac{\mathfrak{C}}{5\cdot 6} + \frac{\mathfrak{D}}{7\cdot 8} - \text{etc.},$$

which series, because of the too strong divergence, is inept to find the value of C at least approximately.

§158 But we will not only find an approximate value, but even the true value of C, if we consider Wallis's expression found for the value of π and demonstrated in the *Introductio*, which was

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot \text{etc.}}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot\ 9\ \cdot 11\cdot 11\cdot \text{etc.}}$$

For, by taking logarithms, it will be

$$\log\pi - \log 2 = 2\log 2 + 2\log 4 + 2\log 6 + 2\log 8 + 2\log 10 + \log 12 + \text{etc.}$$

$$- \log 1 - 2\log 3 - 2\log 5 - 2\log 7 - 2\log 9 - 2\log 11 - \text{etc.}$$

Therefore, in the assumed series let us put $x = \infty$, and since

$$\log 1 + \log 2 + \log 3 + \log 4 + \cdots + \log x = C + \left(x + \frac{1}{2}\right)\log x - x,$$

it will be

$$\log 1 + \log 2 + \log 3 + \log 4 + \cdots + \log 2x = C + \left(2x + \frac{1}{2}\right)\log 2x - 2x$$

and

$$\log 2+\log 4+\log 6+\log 8+\cdots+\log 2x = C+\left(x+\frac{1}{2}\right)\log x - x\log 2 - x,$$

hence

$$\log 1+\log 3+\log 5+\log 7+\cdots+\log(2x-1) = C+\left(x+\frac{1}{2}\right)\log 2 - x.$$

Therefore, since

$$\begin{aligned}\log\frac{\pi}{2} = {}& 2\log 2+2\log 4+2\log 6+\cdots+2\log 2x-\log 2x\\ & -2\log 1\ -2\log 3-2\log 5-\cdots-2\log(2x-1),\end{aligned}$$

having put $x=\infty$, it will be

$$\log\frac{\pi}{2} = 2C+(2x+1)\log x+2x\log 2-2x-\log 2-\log x-2x\log x-(2x+1)\log 2+2x$$

and hence

$$\log\frac{\pi}{2} = 2C-2\log 2, \quad \text{therefore} \quad 2C=\log 2\pi \quad \text{and} \quad C=\frac{1}{2}\log 2\pi,$$

whence in decimal fractions it is found to be

$$C = 0.9189385332046727417803297,$$

and at the same time the following series is summed

$$1-\frac{\mathfrak{A}}{1\cdot 2}+\frac{\mathfrak{B}}{3\cdot 4}-\frac{\mathfrak{C}}{5\cdot 6}+\frac{\mathfrak{D}}{7\cdot 8}-\frac{\mathfrak{E}}{9\cdot 10}+\text{etc.}=\frac{1}{2}\log 2\pi.$$

§159 Now, having found this constant $C=\frac{1}{2}\log 2\pi$, the sum of any number of logarithms from this series $\log 1+\log 2+\log 3+$ etc. can be exhibited. For, if one puts

$$s=\log 1+\log 2+\log 3+\log 4+\cdots+\log x,$$

it will be

$$s = \frac{1}{2}\log 2\pi + \left(x + \frac{1}{2}\right)\log x - x + \frac{\mathfrak{A}}{1 \cdot 2x} - \frac{\mathfrak{B}}{3 \cdot 4x^3} + \frac{\mathfrak{C}}{5 \cdot 6x^5} - \frac{\mathfrak{D}}{7 \cdot 8x^7} + \text{etc.},$$

if the propounded logarithms were hyperbolic; but if common logarithms are propounded, then in the terms $\frac{1}{2}\log 2\pi + \left(x + \frac{1}{2}\right)\log x$ for $\log 2\pi$ and $\log x$ one also has to take common logarithms, but the remaining terms of the series

$$-x + \frac{\mathfrak{A}}{1 \cdot 2x} - \frac{\mathfrak{B}}{3 \cdot 4x^3} + \text{etc.}$$

have to be multiplied by $0.434294481903251827 = n$. Therefore, in this case for common logarithms it will be

$$\begin{aligned} \log \pi &= 0.497149872694133854351268 \\ \log 2 &= 0.301029995663981195213738 \\ \hline \log 2\pi &= 0.798179868358115049565006 \\ \tfrac{1}{2}\log 2\pi &= 0.399089934179057524782503. \end{aligned}$$

Example

Let the aggregate of thousand tabulated logarithms be in question

$$s = \log 1 + \log 2 + \log 3 + \cdots + \log 1000.$$

Therefore, it will be $x = 1000$ and

$$
\begin{array}{lrr}
 & \log x = & 3.000000000000 \\[2ex]
\text{whence} & x \log x = & 3000.000000000000 \\[2ex]
 & \dfrac{1}{2} \log x = & 1.500000000000 \\[2ex]
 & \dfrac{1}{2} \log 2\pi = & 0.399089341790 \\
\cline{3-3}
 & & 3001.8990899341790 \\[2ex]
\text{subtr.} & nx = & 2567.6046080309272
\end{array}
$$

Furthermore,

$$
\begin{array}{lrr}
 & \dfrac{n\mathfrak{A}}{1 \cdot 2x} = & 0.0000361912068 \\[2ex]
\text{subtr.} & \dfrac{n\mathfrak{B}}{3 \cdot 4x^3} = & 0.0000000000012 \\[2ex]
 & \text{add.} = & 2567.6046080309272 \\
\cline{3-3}
\text{sum in question} & s = & 2567.6046442221328.
\end{array}
$$

Therefore, since s is the logarithm of the product of the numbers

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots 1000,$$

it is clear that this product, if actually multiplied, has 2568 digits and the first numbers will be 4023872, which will be followed by 2561 other numbers.

§160 Therefore, by means of this summation of logarithms the products of arbitrary many factors proceeding according to the natural numbers, can be assigned approximately. An application of this is the solution of the problem, in which the middle term or the largest term in any power of the binomial $(a+b)^m$ is in question, where it is certainly to be noted, if m is an odd number, that two equal middle terms are given, which, if they are added, yield the middle term in the following even power. Because hence the largest coefficient in any even power is twice as large as the middle coefficient in the preceding odd power, it will be sufficient to have determined the largest middle term for the even powers. Therefore, let $m = 2n$ and the middle coefficient will be expressed as follows

$$\frac{2n(2n-1)(2n-2)(2n-3)\cdots(n+1)}{1\cdot 2\cdot 3\cdot 4\cdots n}.$$

Let us call this middle coefficient in question $= u$ and one will be able to represent it this way

$$u = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdots 2n}{(1\cdot 2\cdot 3\cdot 4\cdots 2n)^2},$$

and, having taken logarithms, it will be

$$\begin{aligned}\log u &= \log 1 + \log 2 + \log 3 + \log 4 + \log 5 + \cdots + \log 2n\\ &-2\log 1 - 2\log 2 - 2\log 3 - 2\log 4 - 2\log 5 - \cdots - 2\log 2n.\end{aligned}$$

§161 Now, by assuming these logarithms to be hyperbolic logarithms, it will be

$$\log 1 + \log 2 + \log 3 + \log 4 + \cdots + \log 2n = \frac{1}{2}\log 2\pi + \left(2n + \frac{1}{2}\right)\log n + \left(2n + \frac{1}{2}\right)\log 2 - 2n$$
$$+\frac{\mathfrak{A}}{1\cdot 2\cdot 2n} - \frac{\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} + \frac{\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} - \text{etc.}$$

and

$$\begin{aligned}&2\log 1 + 2\log 2 + 2\log 3 + 2\log 4 + \cdots + 2\log n\\ &= \log 2\pi + (2n+1)\log n - 2n + \frac{2\mathfrak{A}}{1\cdot 2n} - \frac{2\mathfrak{B}}{3\cdot 4n^3} + \frac{2\mathfrak{C}}{5\cdot 6n^5} - \text{etc.},\end{aligned}$$

having subtracted which expression from the other it will remain

$$\log u = -\frac{1}{2}\log \pi - \frac{1}{2}\log n + 2n\log 2 + \frac{\mathfrak{A}}{1\cdot 2\cdot 2n} - \frac{\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} + \frac{\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} - \text{etc.}$$

$$-\frac{2\mathfrak{A}}{1\cdot 2n}+\frac{2\mathfrak{B}}{3\cdot 4n^3}-\frac{2\mathfrak{C}}{5\cdot 6n^5}+\text{etc.};$$

by collecting each two terms it will be

$$\log u=\log\frac{2^{2n}}{\sqrt{n\pi}}-\frac{3\mathfrak{A}}{1\cdot 2\cdot 2n}+\frac{15\mathfrak{B}}{3\cdot 4\cdot 2^3n^3}-\frac{63\mathfrak{C}}{5\cdot 6\cdot 2^5n^5}+\frac{255\mathfrak{D}}{7\cdot 8\cdot 2^7n^7}-\text{etc.}$$

Let

$$\frac{3\mathfrak{A}}{1\cdot 2\cdot 2^2n^2}-\frac{15\mathfrak{B}}{3\cdot 4\cdot 2^4n^4}+\frac{63\mathfrak{C}}{5\cdot 6\cdot 2^6n^6}-\frac{255\mathfrak{D}}{7\cdot 8\cdot 2^8n^8}+\text{etc.}$$
$$=\log\left(1+\frac{A}{2^2n^2}+\frac{B}{2^4n^4}+\frac{C}{2^6n^6}+\frac{D}{2^8n^8}+\text{etc.}\right);$$

it will be

$$\log u=\log\frac{2^{2n}}{\sqrt{n\pi}}-2n\log\left(1+\frac{A}{2^2n^2}+\frac{B}{2^4n^4}+\frac{C}{2^6n^6}+\text{etc.}\right)$$

and hence

$$\frac{u^{2n}}{\left(1+\frac{A}{2^2n^2}+\frac{B}{2^4n^4}+\frac{C}{2^6n^6}+\text{etc.}\right)^{2n}\sqrt{n\pi}}.$$

Having put $2n=m$, it will be

$$\log\left(1+\frac{A}{2^2n^2}+\frac{B}{2^4n^4}+\frac{C}{2^6n^6}+\frac{D}{2^8n^8}+\text{etc.}\right)$$

$$\begin{aligned}
&= \frac{A}{m^2}+\frac{B}{m^4}+\frac{C}{m^6}+\frac{D}{m^8}+\frac{E}{m^{10}}+\text{etc.}\\
&\qquad -\frac{A^2}{2m^4}-\frac{AB}{m^6}-\frac{AC}{m^8}-\frac{AD}{m^{10}}-\text{etc.}\\
&\qquad\qquad -\frac{BB}{2m^8}-\frac{BC}{m^{10}}-\text{etc.}\\
&\qquad\qquad +\frac{A^3}{3m^6}+\frac{A^2B}{m^8}+\frac{A^2C}{m^{10}}+\text{etc.}\\
&\qquad\qquad\qquad +\frac{AB^2}{m^{10}}+\text{etc.}\\
&\qquad\qquad\qquad -\frac{A^4}{4m^8}-\frac{A^3B}{m^{10}}-\text{etc.}\\
&\qquad\qquad\qquad +\frac{A^5}{5m^{10}}+\text{etc.};
\end{aligned}$$

since this expression has to be equal to this one

$$\frac{2\mathfrak{A}}{1\cdot 2m^2}-\frac{15\mathfrak{B}}{3\cdot 4}+\frac{63\mathfrak{C}}{5\cdot 6m^6}-\frac{255\mathfrak{D}}{7\cdot 8m^8}+\text{etc.},$$

it will be

$$A = \frac{3\mathfrak{A}}{1 \cdot 2}$$

$$B = \frac{A^2}{2} - \frac{25\mathfrak{B}}{3 \cdot 4}$$

$$C = AB - \frac{1}{3}A^3 + \frac{63\mathfrak{C}}{5 \cdot 6}$$

$$D = AC + \frac{1}{2}B^2 - A^2B + \frac{1}{4}A^4 - \frac{255\mathfrak{D}}{7 \cdot 8}$$

$$E = AD + BC - A^2C - AB^2 + A^3B - \frac{1}{5}A^5 + \frac{1023\mathfrak{E}}{9 \cdot 10}$$

etc.

§162 Since $\mathfrak{A} = \frac{1}{6}$, $\mathfrak{B} = \frac{1}{30}$, $\mathfrak{C} = \frac{1}{42}$, $\mathfrak{D} = \frac{1}{30}$, $\mathfrak{E} = \frac{5}{66}$, it will be

$$A = \frac{1}{4}, \quad B = -\frac{1}{96}, \quad C = \frac{27}{640}, \quad D = -\frac{90031}{2^{11} \cdot 3^2 \cdot 5 \cdot 7} \quad \text{etc.}$$

Therefore, one finds

$$u = \frac{2^{2n}}{\left(1 + \frac{1}{2^4 n^2} - \frac{1}{2^9 \cdot 3n^4} + \frac{27}{2^{13} \cdot 5n^6} - \frac{90031}{2^{19} \cdot 3^2 \cdot 5 \cdot 7n^8} + \text{etc.}\right)^{2n} \sqrt{n\pi}}$$

or

$$u = \frac{2^{2n}\left(1 - \frac{1}{2^4 n^2} + \frac{7}{2^9 \cdot 3n^4} - \frac{121}{2^{13} \cdot 3 \cdot 5n^6} + \frac{107489}{2^{19} \cdot 3^2 \cdot 5 \cdot 7n^8} - \text{etc.}\right)^{2n}}{\sqrt{n\pi}},$$

or if this expansion of the series is actually done, it will approximately be

$$u = \frac{2^{2n}}{\sqrt{n\pi}\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^5} - \frac{5}{16 \cdot 128n^4} + \text{etc.}\right)};$$

therefore, the middle term in $(1+1)^{2n}$ will have the same ratio to the sum of all terms 2^{2n}

$$\text{as} \quad 1 \quad \text{to} \quad \sqrt{n\pi\left(1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{1}{128n^3} - \frac{5}{16 \cdot n^4} + \text{etc.}\right)};$$

or, for the sake of brevity having put $4n = \nu$, this ratio will be

$$\text{as} \quad \text{to} \quad \sqrt{n\pi\left(1 + \frac{1}{\nu} + \frac{1}{2\nu^2} - \frac{1}{2\nu^3} - \frac{5}{8\nu^4} + \frac{23}{8\nu^5} + \frac{53}{16\nu^6} - \text{etc.}\right)}.$$

Example 1

Let the middle term in the expanded binomial $(a+b)^{10}$ be in question, which is known to be

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 252.$$

Applying the last formula found for u it will be $n = 5$ and hence

$$\begin{array}{lrl}
 & \dfrac{1}{4n} & = 0.0500000 \\
 & \dfrac{1}{32n^2} & = 0.0012500 \\
 & & \quad \overline{0.0512500} \\
\text{subtract} & \dfrac{1}{128n^3} & = 0.0000625 \\
 & & \quad \overline{0.0511875} \\
\text{Therefore} & 1 + \dfrac{1}{4n} + \text{etc.} & = 1.0511836 \\
\text{the log. of this} & & = \overline{0.0216784} \\
 & \log n & = 0.6989700 \\
 & \log \pi & = 0.4971498 \\
 & & \quad \overline{1.2177982} \\
 & \log \sqrt{n\pi(1+\text{etc.})} & = 0.6088991 \\
\text{from} & \log 2^{2n} & = 3.0102999 \\
 & \log u & = \overline{2.4014008} \\
\text{whence} & u & = 252.
\end{array}$$

EXAMPLE 2

Investigate the ratio which the middle term has to sum of all terms in the hundredth power of the binomial $1+1$, *which sum is* 2^{100}.

For this, let us use the formula found first

$$\log u = \log \frac{2^{2n}}{\sqrt{n\pi}} - \frac{3\mathfrak{A}}{1\cdot 2\cdot 2n} + \frac{15\mathfrak{B}}{3\cdot 4\cdot 2^3 n^3} - \frac{63\mathfrak{C}}{5\cdot 6\cdot 2^5 n^5} + \text{etc.},$$

in which, having put $2n = m$, that one has this power $(1+1)^m$ and having substituted the values for $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., it will be

$$\log u = \log \frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{1}{4m} + \frac{1}{24m^3} - \frac{1}{20m^5} + \frac{17}{112m^7} - \frac{31}{36m^9} + \frac{691}{88m^{11}} - \text{etc.};$$

since these logarithms are hyperbolic, multiply them by

$$k = 0.434294481903251,$$

that they are transformed into tabulated ones, and it will be

$$\log u = \log \frac{2^m}{\sqrt{\frac{1}{2}m\pi}} - \frac{k}{4m} + \frac{k}{24m^3} - \frac{k}{20m^5} + \frac{17k}{112m^7} - \frac{31k}{36m^9} + \text{etc.},$$

whence, because the middle term is u, the ratio in question will be $2^m : u$ and hence

$$\log \frac{2^m}{u} = \log \frac{1}{2}m\pi + \frac{k}{4m} - \frac{k}{24m^3} + \frac{k}{20m^5} - \frac{17k}{122m^7} + \frac{31k}{36m^9} - \frac{691k}{88m^{11}} + \text{etc.}$$

Therefore, since, because of the exponent $m = 100$,

$$\frac{k}{m} = 0.0043429448, \quad \frac{k}{m^3} = 0.0000004343, \quad \frac{k}{m^5} = 0.0000000000,$$

it will be

$$\begin{aligned} \frac{k}{4m} &= 0.0010857362 \\ \frac{k}{24m^3} &= 0.0000000181 \\ \hline & 0.0010857181 \end{aligned}$$

Then

$$\begin{aligned} \log \pi &= 0.4971498726 \\ \log \frac{1}{2}m &= 1.6989700043 \\ \hline \log \frac{1}{2}m\pi &= 2.1961198769 \\ \hline \log \sqrt{\frac{1}{2}m\pi} &= 1.0980599384 \\ \frac{k}{4m} - \frac{k}{24m^3} + \text{etc.} &= 0.0010857181 \\ \hline & 1.0991456565 = \log \frac{2^{100}}{u}. \end{aligned}$$

Therefore, it will be $\frac{2^{100}}{u} = 12.56451$ and hence in the expanded power $(1+1)^{100}$ the middle term will have the same ratio to the sum of all terms 2^{100} as 1 to 12.56451.

§163 Now, let the general term z denote the exponential function a^x so that this geometric series has to be summed

$$s = a + a^2 + a^3 + a^4 + \cdots + a^x;$$

since it is a geometric series, its sum is already known; for, it will be $s = \frac{(a^x - 1)a}{a-1}$. But let us investigate this sum in the way explained here. Since $z = a^x$, it will be $\int z dx = \frac{a^x}{\log a}$; for, the differential of this is $a^x dx$; but then it will be

$$\frac{dz}{dx} = a^x \log a, \quad \frac{ddz}{dx^2} = a^x (\log a)^2, \quad \frac{d^3 z}{dx^3} = a^x (\log a)^3 \quad \text{etc.},$$

whence it follows that

$$s = a^x \left(\frac{1}{\log a} + \frac{1}{2} + \frac{\mathfrak{A}}{1 \cdot 2} \log a - \frac{\mathfrak{B}}{1 \cdot 2 \cdot 3 \cdot 4} (\log a)^3 + \frac{\mathfrak{C}}{1 \cdot 2 \cdot 3 \cdots 6} (\log a)^5 - \text{etc.} \right) + C.$$

To define the constant C put $x = 0$ and, because of $s = 0$, it will be

$$C = -\frac{1}{\log a} - \frac{1}{2} - \frac{\mathfrak{A}}{1 \cdot 2} \log a + \frac{\mathfrak{B}}{1 \cdot 2 \cdot 3 \cdot 4} (\log a)^3 - \text{etc.}$$

and hence it will be

$$s = (a^x - 1) \left(\frac{1}{\log a} + \frac{1}{2} + \frac{\mathfrak{A}}{1 \cdot 2} \log a - \frac{\mathfrak{B}}{1 \cdot 2 \cdot 3 \cdot 4} (\log a)^3 + \frac{\mathfrak{C}}{1 \cdot 2 \cdot 3 \cdots 6} (\log a)^5 - \text{etc.} \right)$$

Therefore, because the sum is $\frac{(a^x - 1)a}{a-1}$, it will be

$$\frac{a}{a-1} = \frac{1}{\log a} + \frac{1}{2} + \frac{\mathfrak{A}}{1 \cdot 2} \log a - \frac{\mathfrak{B}}{1 \cdot 2 \cdot 3 \cdot 4} (\log a)^3 + \frac{\mathfrak{C}}{1 \cdot 2 \cdot 3 \cdots 6} (\log a)^5 - \text{etc.},$$

where $\log a$ denotes the hyperbolic logarithm of a; therefore,

$$\frac{(a+1) \log a}{2(a-1)} = 1 + \frac{\mathfrak{A}(\log a)^2}{1 \cdot 2} - \frac{\mathfrak{B}(\log a)^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{\mathfrak{C}(\log a)^5}{1 \cdot 2 \cdot 3 \cdots 6} - \text{etc.}$$

and so one will be able to exhibit the sum of this series.

§164 Let the general term be $z = \sin ax$ and

$$s = \sin a + \sin 2a + \sin 3a + \cdots + \sin ax;$$

this series, since it is recurring, can also be summed; for, it will be

$$s = \frac{\sin a + \sin ax - \sin(ax + a)}{1 - 2\cos a + 1} = \frac{\sin a + (1 - \cos a) \sin ax - \sin a \cdot \cos ax}{2(1 - \cos a)}.$$

It will be

$$\int z dx = \int dx \sin ax = -\frac{1}{a} \cos ax$$

and

$$\frac{dz}{dx} = a\cos ax, \quad \frac{d^3z}{dx^3} = -a^3\cos ax, \quad \frac{d^5z}{dx^5} = a^5\cos ax \quad \text{etc.}$$

Therefore,

$$s = C - \frac{1}{a}\cos ax + \frac{1}{2}\sin ax + \frac{\mathfrak{A}a\cos ax}{1\cdot 2} + \frac{\mathfrak{B}a^3\cos ax}{1\cdot 2\cdot 3\cdot 4}$$
$$+\frac{\mathfrak{C}a^5\cos ax}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \frac{\mathfrak{D}a^7\cos ax}{1\cdot 2\cdots 8} + \text{etc.}$$

Put $x = 0$ so that $s = 0$, and it will be

$$C = \frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6} - \text{etc.},$$

therefore

$$s = \frac{1}{2}\sin ax + (1-\cos sx)\left(\frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6} - \text{etc.}\right).$$

But since

$$s = \frac{1}{2}\sin ax + \frac{(1-\cos ax)\sin a}{2(1-\cos a)},$$

it will become

$$\frac{\sin a}{2(1-\cos a)} = \frac{1}{2}\cot\frac{1}{2}a = \frac{1}{a} - \frac{\mathfrak{A}a}{1\cdot 2} - \frac{\mathfrak{B}a^3}{1\cdot 2\cdot 3\cdot 4} - \frac{\mathfrak{C}a^5}{1\cdot 2\cdots 6} - \text{etc.},$$

which same series we already found above (§ 127).

§165 Now let $z = \cos ax$ and the series to be summed

$$s = \cos a + \cos 2a + \cos 3a + \cdots + \cos ax;$$

the sum of this series, since it is recurring, will be

$$s = \frac{\cos a - 1 + \cos ax - \cos(ax+a)}{1 - 2\cos a + 1} = -\frac{1}{2} + \frac{1}{2}\cos ax + \frac{1}{2}\cot\frac{1}{2}a\cdot\sin ax.$$

But on the other hand to express the sum by means of our method note that it will be

$$\int z dx = \int dx \cos ax = \frac{1}{a} \sin ax$$

and

$$\frac{dz}{dx} = -a \sin ax, \quad \frac{d^3 z}{dx^3} = a^3 \sin ax, \quad \frac{d^5 z}{dx^5} = -a^5 \sin ax \quad \text{etc.}$$

Therefore,

$$s = C + \frac{1}{a} \sin ax + \frac{1}{2} \cos ax - \frac{\mathfrak{A} a \sin ax}{1 \cdot 2} - \frac{\mathfrak{B} a^3 \sin ax}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}$$

Let $x = 0$, it will be $s = 0$ and $C = -\frac{1}{2}$ and hence it will be

$$s = -\frac{1}{2} + \frac{1}{2} \cos ax + \frac{1}{a} \sin ax - \frac{\mathfrak{A} a \sin ax}{1 \cdot 2} - \frac{\mathfrak{B} a^3 \sin ax}{1 \cdot 2 \cdot 3 \cdot 4} - \text{etc.}$$

Therefore, since

$$s = -\frac{1}{2} + \frac{1}{2} \cos ax + \frac{1}{2} \cot \frac{1}{2} a \cdot \sin ax,$$

it will be, as we just found [§ 164],

$$\frac{1}{2} \cot \frac{1}{2} a = \frac{1}{a} - \frac{\mathfrak{A} a}{1 \cdot 2} - \frac{\mathfrak{B} a^3}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{\mathfrak{C} a^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \text{etc.}$$

§166 Since we found above [§ 92], if a denotes any arc, that

$$\frac{\pi}{2} = \frac{a}{2} + \sin a + \frac{1}{2} \sin 2a + \frac{1}{3} \sin 3a + \frac{1}{4} \sin 4a + \text{etc.},$$

let us consider this series and let $z = \frac{1}{x} \sin ax$ that

$$s = \sin a + \frac{1}{2} \sin 2a + \frac{1}{3} \sin 3a + \cdots + \frac{1}{x} \sin ax.$$

But in this case $\int z dx = \int \frac{dx}{x} \sin ax$, which integral cannot be exhibited. Therefore, it will be

$$\frac{dz}{dx} = \frac{a}{x} \cos ax - \frac{1}{xx} \sin ax, \quad \frac{ddz}{dx^2} = -\frac{a^2}{x} \sin ax - \frac{2a}{xx} \cos ax + \frac{2}{x^3} \sin ax,$$

$$\frac{d^3z}{dx^3} = -\frac{a^3}{x}\cos ax + \frac{3a^2}{x^2}\sin ax + \frac{6a}{x^3}\cos ax - \frac{6}{x^4}\sin ax,$$
$$\frac{d^4z}{dx^4} = \frac{a^4}{x}\sin ax + \frac{4a^3}{xx}\cos ax - \frac{12a^2}{x^3}\sin ax - \frac{24a}{x^4}\cos ax + \frac{24}{x^5}\sin ax.$$

Therefore, since neither the integral formula $\int z dx$ can be exhibited nor it is possible to express these differentials conveniently, we are not able to define the sum of this series by means of this expression, such that anything could be concluded from this. The same inconvenience occurs in many other series, if the general term is not simple enough that its differentials can be expressed in general. But in the following chapter we will find other general expressions for the sums of the series whose general terms are either too composite or cannot be given at all; these can be applied successfully. But the insufficiency of the method treated here is especially revealed, if the signs of the terms of the propounded series alternate; for, then, even though the general terms are simple, the summatory terms can nevertheless not be expressed conveniently using this method.